Some geometric features of Berry´s phase

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February 1, 2008

Abstract

In this letter, we elaborate on the identification and construction of the differential geometric elements underlying Berry´s phase. Berry bundles are built generally from the physical data of the quantum system under study. We apply this construction to typical and recently investigated systems presenting Berry´s phase to explore their geometric features.

Berry´s phase [1] discovery for parameter-dependent quantum systems showed the existence of fundamental differential geometric features in quantum physics. In fact, the *geometric nature* of this phase leads to both, its theoretical importance and the ability to perform experiments in which this phase can be detected [2]. Because of this fact, it is important to have a suitable description of the underlying physical data of the system in terms of the geometric objects which lead to the phase shift under study.

Usually [3] for *non abelian* phases, this geometric setting is modeled by means of the universal $U(k)$ bundle over the grassmannian manifold $G_{K^m}(\mathcal{H})$ [4] of K^m−dimensional subspaces of the total Hilbert space $\mathcal{H}$, endowed with its canonical connection. When the parameter b varies within a parameter space $\tilde{B}$, the K^m−dimensional *eigenspace* $F_b^m \subset \mathcal{H}$ of a given *energy* $\epsilon^m(b)$ also varies describing a curve in the grassmannian. Parallel transport along this curve captures the geometric Berry phase effect.

The aim of this letter is to enlighten the fact that the geometry directly relevant for the study of the underlying physical problem is not that of the above mentioned universal bundle, but the one of the *pull back* [4] bundle along the induced map Parameter Space⟶Grassmannian. Indeed, the space of *physical parameters* can be much smaller than or have a very different geometry from that of the grassmannian manifold. We remark this in the same sense that the geometry of a 2−sphere $S^2 \hookrightarrow \mathbb{R}^3$, even though following form that of $\mathbb{R}^3$, is different and can be independently studied from that of the bigger ambient space $\mathbb{R}^3$. A very simple example of this is given by a large ($s \geq 1$) spin s in an external magnetic field. In that case, the dimension of the grassmanninan can be huge (equal to $4s$) while the space of physically accessible eigenspaces $F_b^m \subset \mathcal{H}$ through the manipulation of the external magnetic field, is a submanifold of at most dimension 3 for every s (see examples below).

On the other hand, considering the pull back bundle has the strategical advantage of allowing for a direct study on how the geometry of the physical parameter space B affects the resulting Berry phase. Moreover, this pull back bundle $U(E^m) \longrightarrow B$, that we shall refer to as *Berry bundle*, can be directly constructed from the *natural physical inputs* defining the quantum system:

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the Hilbert space of states $\mathcal{H}$ and the parameter dependent Hamiltonian $H(b)$; with no further reference to the universal bundle. The construction itself yields the relevant topology of the *allowed parameter space* $B \subset \tilde{B}$ ([8]). In the next sections, we give the details of this basic and direct construction of the Berry bundle, which captures the essential geometric features of the parameter dependence of the system. In this context, the universal bundle has an *a posteriori* appearance due only to its universality property [4].

Finally, we remark that this construction has also the advantage of *computability*: in most cases, one can concretely build and characterize the Berry bundle together with the corresponding *Berry connection* on it. The reason is that, typically, the allowed parameter space B, which is the base of the Berry bundle, is just $\mathbb{R}^n$ minus *singular points*. Consequently, a direct characterization of the bundle in terms of an open cover and transition functions becomes doable.

The reader might also find interesting the fact that the above mentioned basic ingredients of bundle theory, v.g. transition functions and local connection expressions, seem to be hand-tailored for the sole study of Berry phases.

At the end of this letter, as suggested in [5], we apply the mentioned construction to obtain new results on the global geometry underlying typical, and recently investigated, concrete quantum systems presenting Berry phase effects.

Setting and notation.- We now recall some well known facts about (non abelian) Berry´s phases. Let $(\mathcal{H}, H(b))$ be a *Quantum system*, where the *Hamiltonian* operator $H(b) : \mathcal{H} \longrightarrow \mathcal{H}$ depends smoothly on *parameters* $b \in \tilde{B}$, for $\tilde{B}$ a manifold from where, a priori, the parameters can be chosen.

For each parameter b, $P^m(b) : \mathcal{H} \longrightarrow F_b^m$ denotes the corresponding *orthogonal projection*. As usual, the *evolution operator* $U_{t,t_0} : \mathcal{H} \longrightarrow \mathcal{H}$ is a unitary operator s.t. $\psi(t) = U_{t,t_0}\psi(t_0)$, for $\psi(t)$ denoting the state of the system at time t. Consider $b(t) : I = [t_i, t_f] \longrightarrow \tilde{B}$ a piece-wise smooth curve on the *parameter space* $\tilde{B}$. If the time evolution of the parameters $b(t)$ is *slow*, we can assume that this evolution is *adiabatic*, and thus it is a good approximation to assume that if $\psi(t_0) \in F_{b(t_0)}^m$ then $\psi(t) \in F_{b(t)}^m$ for each $t \in I$. Consequently, the evolution operator can be written as[6] $U_{t,t_0} = \sum_m U_{t,t_0}^m$, where the blocks are linear (unitary) maps $U_{t,t_0}^m = P^m(b(t))U_{t,t_0}P^m(b(t_0)) : F_{b(t_0)}^m \longrightarrow F_{b(t)}^m$.

Consider, as usual, a curve $R_{t,t_0} : I \longrightarrow U(\mathcal{H})$ of unitary linear operators in $\mathcal{H}$ taking the eigenspace at time t_0, $F_{b(t_0)}^m$, to the eigenspace at time t, $F_{b(t)}^m$. If we write $U_{t,t_0}^m = R_{t,t_0}\tilde{U}_{t,t_0}^m = R_{t,t_0}P^m(b(t_0))\tilde{U}_{t,t_0}^m$ where $\tilde{U}_{t,t_0}^m \in U(F_{b(t_0)}^m)$ is a linear unitary endomorphism of the subspace $F_{b(t_0)}^m \in \mathcal{H}$, it follows from Schrodinger equation that $\tilde{U}_{t,t_0}^m = U_{dyn}^m(t)U_{geom}^m(t)$, the product of the *dynamical* and *geometrical* part of the solution. The (non abelian) geometric part is defined by

$$\frac{d}{dt}U_{geom}^m(t)U_{geom}^{m\dagger}(t) = -P^m(b(t_0))R_{t,t_0}^{-1}\frac{d}{dt}R_{t,t_0}P^m(b(t_0)). \tag{0.1}$$

When the parameter space curve $b(t)$ is closed , i.e. $b(t_0) = b(t_f)$, then the unitary operator $U_{geom}^m(t_f)$ is the (non-abelian) geometric **Berry phase** (factor) for the underlying adiabatic parameter-dependent quantum system. In what follows, the geometric nature of $U_{geom}(t_f)$ shall become clear.

Constructing the Berry bundles over parameter space.- We shall now construct a vector bundle E^m over the *allowed parameter space* $B \subset \tilde{B}$, capturing the essential geometry underlying the parameter dependence of the system. This construction can be straightforwardly adapted to the case of adiabatic and invariant operators as suggested in [7]. This bundle must have as fiber over each parameter $b \in B$, the vector space of all possible eigenstates of $H(b)$ with

eigenvalue $\epsilon^m(b)$. For, under an *adiabatic* change of parameters $b(t)$, the system evolution will be described by a curve $\psi(t)$ in E^m projecting onto $b(t)$ in the base B. To begin the construction, we assume some smoothness conditions on the parameter dependence of H:

(*i*) we can smoothly (in b) choose the eingenvalues $\epsilon^m(b)$ of $H(b)$, $m = 1, 2, ...$ via smooth maps $B \longrightarrow \mathbb{R}$ taking $b \longmapsto \epsilon^m(b)$ such that $det(H(b) - \epsilon^m(b)id_{\mathcal{H}}) = 0$ for all $b \in B$.

(*ii*) the degeneracy degree $K_m := dim(F_b^m)$ of the energy level $\epsilon^m(b)$ is constant for all $b \in B$.

The space of *allowed parameters* B is just $\tilde{B}$ minus singular points where the above conditions do not hold. Removing singular points creates holes yielding a non trivial topology on B and allowing for non trivial bundles (underlying Berry´s phase) over it[8]. Note that this parameter space topology arises naturally within our construction from the physical inputs.

Vector Berry bundle $E^m \longrightarrow B$.- Fix an energy label m and consider the map $\mathfrak{F}^m$: $B \times \mathcal{H} \longrightarrow B \times \mathcal{H}$ as $\mathfrak{F}^m(b, \alpha) := (b, H(b)\alpha - \epsilon^m(b)\alpha)$. This is a vector bundle morphism and, since the dimension of the eigenspaces of $H(b)$ is assumed to be constant, the rank of $\mathfrak{F}$ is the same on all fibers. So $E^m := Ker(\mathfrak{F}^m)$ is also a vector bundle over B which we shall refer to as the *vector Berry bundle*. As mentioned in the introduction, this vector bundle can be also obtained from a canonical one by pull back via the map $B \longrightarrow G_{K^m}(\mathcal{H})$. Note that, as desired, $E^m = Ker(\mathfrak{F}^m) = \underset{b \in B}{\sqcup} F_b^m$ and the projection is given by $E^m \xrightarrow{\pi} B$, $|m(b), i\rangle \longmapsto b$ when $|m(b), i\rangle \in F_b^m$, $i = 1...K_m$. Moreover, the vector bundle E^m is endowed with the **fiber metric** induced by that of the Hilbert space $\mathcal{H}$. We stress that the geometry of this bundle is determined directly by the physical data involving the dependence of the Hamiltonian $H(b)$ on the parameters b.

The principal $U(K_m)-$Berry bundle $U(E^m)$ over B.- We now perform the analogue of passing to describe the system with the evolution operator in $U(\mathcal{H})$ instead of using time dependent states in $\mathcal{H}$. Moreover, the construction we give below can be applied to any Hermitian vector bundle E^m over a smooth manifold B, showing that the relevant geometric data of the problem is already encoded in E^m. Consider the $U(K_m)$-principal bundle of *orthonormal bases* $\{|m(b), i\rangle\}$ of E^m; $\pi : U(E^m) \longrightarrow B$, $\{|m(b), i\rangle\} \mapsto b$ on which the Lie group $U(K_m)$ acts on the *right* via $\{|m(b), k\rangle\} \cdot (a_j^i) = \{\widetilde{|m(b), k\rangle} = \Sigma_j \ a_j^k |m(b), j\rangle\}$. We shall refer to $U(E^m) \longrightarrow B$ as the *Berry Bundle* over the allowed parameter space B. As stated in the introduction, this principal bundle $U(E^m)$ can be obtained from the corresponding universal one by *pulling back* via the function $B \longrightarrow G_{K^m}(\mathcal{H})$. Choosing a $b_0 \in B$ and using the bijection $U(E^m) \equiv \underset{b}{\bigsqcup} \{linear \ maps \ \hat{u} : F_{b_0}^m \longrightarrow F_b^m \ such \ that \ \langle \hat{u}v, \hat{u}w \rangle_{\mathcal{H}} = \langle v, w \rangle_{\mathcal{H}} \}$, every element $\{|m(b), k\rangle\} \in U(E^m)$ can be seen as a map $\hat{u} : F_{b_0}^m \longrightarrow F_b^m$.

Local and global geometries.- It can be seen[4] that, for every $b_0 \in B$, there exists an *open patch* $U_{b_0} \subseteq B$ of b_0 and a smooth map

$$R : U_{b_0} \longrightarrow Isom(F_{b_0}^m \longrightarrow \mathcal{H}) \tag{0.2}$$

such that, for each $b \in U_{b_0}$, $\{R_b |m(b_0), i\rangle\}_{i=1}^{K_m}$ is a (*moving*) *orthonormal basis* of the eigenspace above parameter b, $F_b^m \subseteq \mathcal{H}$. Above $\{|m(b_0), i\rangle\}_{i=1}^{K_m}$ is a fixed orthonormal basis of $F_{b_0}^m$ and $Isom(F_{b_0}^m \longrightarrow \mathcal{H})$ denotes the manifold of linear isometries from F into $\mathcal{H}$. The map R_b of eq. (0.2) defines smooth **local sections** of the Berry bundles E^m and $U(E^m)$ by $\sigma_i^R(b) = R_b |m(b_0), i\rangle$ and $\Sigma^R(b) = R_b P^m(b_0) : F_{b_0}^m \longrightarrow F_b^m$, respectively. Note, however, that the bundles might not admit *global sections* because, in the intersection of patches, different moving basis of F_b^m might be glued together in a nontrivial fashion. This *global geometry*, fixed by the b

dependence of the system´s Hamiltonian $H(b)$, is captured by the Berry bundle´s the *transition functions*[9] $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \subset B \longrightarrow U(K_m) \simeq U(F_{b_0}^m)$ given by

$$\psi_{\alpha\beta}(\ b) = \sum_{i,j=1}^{K_m} \left\langle w_\beta^j(b) | w_\alpha^i(b) \right\rangle \left(\left| w^j(b_0) \right\rangle \left\langle w^i(b_0) \right| \right) \tag{0.3}$$

where $\left| w_{\alpha,\beta}^i(b) \right\rangle = R_b^{\alpha,\beta} \left| w^i(b_0) \right\rangle$ for $R_b^{\alpha,\beta} : U_{\alpha,\beta} \longrightarrow U(E^m)$ local sections on two intersecting patches U_α and U_β and $\left\{ \left| w^i(b_0) \right\rangle \right\}$ a fixed o.n. basis for the fiber $F_{b_0}^m$ over a chosen $b_0 \in B$.

Berry connection giving the geometric phase.- From eq. (0.2), take $R_{t,t_0} P^m(b_0)$ as $R_{b(t)} \in U(E^m)$ and $U_{geom}^m(t) \in U\left(F_{b_0}^m \right) \simeq U\left(\mathbb{C}^{K_m} \right)$ to be determined. It can be easily seen that there exists a globally defined **principal connection**[9] $A^m : TU(E^m) \longrightarrow u(K_m) = Lie(U(K_m))$, that we shall call the **Berry connection**, on the principal fiber bundle $U(E^m) \overset{\pi}{\longrightarrow} B$ such that its *local expression* along a section Σ^R is

$$(\Sigma^{R*} A^m)_b = P^m(b_0) \ \ R^{-1}(b)\big|_{F_b^m} d_b R \ P^m(b_0)$$

$$= \sum_{k=1}^{dimB} \sum_{i,j=1}^{K_m} \left\langle w^j(b) | \frac{\partial}{\partial b_k} | w^i(b) \right\rangle \left(\left| w^j(b_0) \right\rangle \left\langle w^i(b_0) \right| \right) \ db_k, \tag{0.4}$$

where where b_k are local coordinates on $U_{b_0} \subset B$.

Then, eq. (0.1) for the geometric phase is equivalent to requiring the curve $R_{t,t_0} P^m(b_0) \cdot U_{geom}^m(t)$ in the Berry bundle $U(E^m)$ to be *horizontal* with respect to the Berry connection. Consequently, $R_{t,t_0} P^m(b_0) \cdot U_{geom}^m(t)$ is the *parallel transport* of the initial condition $id_\mathcal{H} P^m(b_0) \cdot U_{geom}^m(t_0)$ along $b(t) \in B$. When the parameter curve is closed $b(t_0) = b(t_f)$ and $U_{geom}^m(t_0) = id_{F_{b_0}^m}$, parallel transport from patch to patch yields a global and geometrically defined *holonomy*[9] $U_{geom}^m(t_f) \in U\left(F_{b_0}^m \right)$ which is precisely the (non abelian if $K_m > 1$) Berry phase[1] factor associated to the underlying quantum system.

Geometry of Berry bundles for $B \simeq S^{2,1}$.- The present approach allows for the use of the physical data encoded in the topology of the parameter space B of the system under study. Indeed, assuming that B is (smooth) homotopically equivalent to the sphere $S^{k=1,2}$, permits to yield conclusions on the geometry of the Berry bundle for a general $H(b)$, by means of some bundle-theoretic results.

(I) For $k = 2$ it holds that if $E^m \longrightarrow B$ is orientable as a vector bundle, then the principal $U(K_m)$-bundle $U(E^m) \longrightarrow B$ is trivializable.

(II) For $k = 1$, then $U(E^m) \longrightarrow B$ is always trivializable.

It is clear that, when the Berry bundle is trivializable, local considerations extend to the whole parameter space B by the existence of smooth *global* sections. This simplifies the analysis and shows that there is no *nontrivial geometric contribution* to the GP.

Spin in magnetic field.- Below, we elaborate on the global geometry underlying the example presented by Berry [1]. Let $\mathcal{H} = V_{spin(s)}$ be the Hilbert space corresponding to a particle with spin s. For $b \in \tilde{B} := \mathbb{R}^3$, let $H(b) = g \hbar b \cdot \hat{s}$ be the Hamiltonian giving the interaction between this particle and a magnetic field represented by b. For a fixed $b \in \mathbb{R}^3$, the $(2s + 1)$ eingenvalues of $H(b)$ are $\epsilon^m(b) = g \hbar \|b\| m$, all with degeneracy 1 except for $b = \mathbf{0}$ where all eigenvalues collapse to only one with full degeneracy. We thus see that the largest submanifold $B \subset \mathbb{R}^3$ satisfying condition (ii) above is $B := \mathbb{R}^3 - \{\mathbf{0}\}$ which, topologically, is equivalent to a $2-$sphere. Which is the Berry bundle for this spin systems? Below we answer this question.

Fix m s.t. $-s \leq m \leq s$. Since $B \approx \mathbb{R}_{>0} \times S^2$, we can cover B with two open sets $U^{\pm} :=$ $\mathbb{R}_{>0} \times \left(S^2 - \{b_0^{\pm}\}\right)$ over which the Berry bundles are trivial. Indeed, if $b_0^{\pm} = (0,0,\pm 1) \in S^2$ denote the poles, each $b \in U^{\pm}$ can be taken by a rotation to the pole. This rotation induces a spin rotation $\hat{R}^{\pm}(b)$ taking the eigenvector $\left|b_0^{\pm}, m\right\rangle \in F_{b_0}^m$ to $|b^{\pm}, m\rangle \in F_b^m$ and can be smoothly chosen for each b defining local sections on $U^{\pm}$ as in (0.2)[11].

The *isomorphism class*[4] of the $U(1)$ principal Berry bundle $U(E^m)$ over B for this system is thus determined by the *homotopy class* of the transition function restricted to the where the two patches intersect, i.e., by $\psi_{+-} : \mathbb{R}_{>0} \times \{equator\ of\ S^2\} \longrightarrow U(1) \simeq \pi^{-1}(b_0^+) = \{\hat{u} : E_{b_0^+}^m \circlearrowleft\}$. A straightforward calculation of the *transition function* (0.3) at $b = \|b\| \left(cos\varphi, sin\varphi, 0\right) \in \mathbb{R}_{>0} \times \{equator\ of\ S^2\}$ yields

$$\psi_{+-}(b) = e^{i2m\varphi} \left\langle b_0^+, m | b_0^-, m \right\rangle \left|b_0^+, m\right\rangle \left\langle b_0^+, m\right|.$$

Since we can take $\left|b_0^{\pm}, m\right\rangle = |\pm m\rangle$ with $\hat{S}_z |\pm m\rangle = \pm m |\pm m\rangle$ and, since $\left\langle b_0^+, m | b_0^-, m\right\rangle$ never vanishes, we conclude that the transition map ψ_{+-} *winds* the equator of S^2 $2m-$times in $U(1)$. This winding number *specifies the above mentioned isomorphism class of the bundle* $U(E^m)$. Note that the homotopy class of the transition map above does not depend on $\|b\|$, as expected. Moreover, this class depends *only* on the $z-$axis spin projection m and *not on*

the total spin s. From this follows the fundamental result that, for particles of spin s and s' such that m is an allowed value of the $z-$axis spin projection for both, then $E^{m,s} \simeq E^{m,s'}$ and $U(E^{m,s}) \simeq U(E^{m,s'})$. For example, any fermion (boson) shall yield the same $U(E^{m=\frac{1}{2}})$ $(U(E^{m=0}))$ modulo isomorphism. Concretely, for $m = \frac{1}{2}$, the transition function is characterized by the winding number 1, and so the Berry bundle is isomorphic to the well known *Hopf fibration* $SU(2) \approx S^3 \longrightarrow S^2$[4]. For $m = 1$, the winding number is 2, and then the Berry bundle is isomorphic to the bundle $SO(3) \longrightarrow S^2$, $r \longmapsto r \cdot b_0^+$ related to classical *rigid body geometric phases*[12].

From the local expression (0.4) for the Berry connection on U^+, it can be easily seen that the holonomy $Hol(\Gamma) \in U(1) \simeq \{\hat{u} : E_{b_0^+}^m \circlearrowleft\}$ of any simple closed path Γ in B (measured from the identity of $U(1)$) reads $Hol(\Gamma) = exp(-im\Omega(\Gamma)) \left|b_0^+, m\right\rangle \left\langle b_0^+, m\right|$, where $\Omega(\Gamma)$ is a (signed) solid angle having Γ as boundary. We thus recover the expression first obtained by Berry [1].

Non abelian phase from holonomic quantum computation.- We take the following example from [10] in the context of *holonomic quantum computation*. Non abelian Berry phases play a crucial role in the theoretical construction of fault tolerant quantum gates because of their *geometric nature*. The setting is the following: $\mathcal{H} = Span\{|0\rangle, |1\rangle, |a\rangle, |e\rangle\}$ and $H(\overrightarrow{\Omega}) = |e\rangle \left(\Omega_0 \langle 0| + \Omega_1 \langle 1| + \Omega_a \langle a|\right) + h.c.$ for $\overrightarrow{\Omega} = (\Omega_0, \Omega_1, \Omega_a) \in \mathbb{R}^3$ being the associated Rabbi frequencies.

The eigenvalues are $\epsilon^0(\overrightarrow{\Omega}) = 0$ and $\epsilon^{\pm}(\overrightarrow{\Omega}) = \pm \left\|\overrightarrow{\Omega}\right\|$. We see that for $\overrightarrow{\Omega} \neq 0$, conditions $(i)(ii)$ given above are satisfied and that $dim E_{\overrightarrow{\Omega}}^0 = 2 = const$. So we take $B = \mathbb{R}^3 - \{0\}$ as in the previous example. Note that in this case, since $dim E_{\overrightarrow{\Omega}}^0 = 2$, the structure group of the associated Berry bundle $U(E^0) \longrightarrow B$ is $U(2)$ and, thus, the induced Berry phase shall be *non abelian*.

Identifying the $\mathbb{R}^3$ vector components $\check{1} \equiv \check{x}$, $\check{0} \equiv \check{y}$, $\check{a} \equiv \check{z}$, it is easy to define local sections of the bundle $E^0 \longrightarrow B$ in $U^{\pm}$ as in the previous example. Also as before, we calculate the transition function $\psi_{+-} : \mathbb{R}_{>0} \times \{equator\ of\ S^2\} \longrightarrow U(2) \simeq \pi^{-1}(\vec{\Omega}_0^+) = \{\hat{u} : E_{\vec{\Omega}_0^+}^0 \circlearrowleft\}$ which determines the

global geometry of the Berry bundle. For $\vec{\Omega} = \left\|\vec{\Omega}\right\| \cdot (sin\alpha, cos\alpha, 0) \in \mathbb{R}_{>0} \times \{equator\ of\ S^2\}$[13]

$$\psi_{+-}(\alpha) \equiv \begin{pmatrix} -cos2\alpha & -sin2\alpha \\ -sin2\alpha & cos2\alpha \end{pmatrix} \in U(2)$$

for $\alpha \in S^1$ parameterizing the equator of S^2. The above $U(2)$ matrix is the product of a constant one by an α-dependent matrix staying within $SU(2) \subset U(2)$ for all α. From this, it follows that the vector Berry bundle $E^0 \longrightarrow B$ is orientable. Thus, by result (I) given above, the $U(2)$ bundle $U(E^0) \longrightarrow B$ is *isomorphic to the trivial bundle $B \times U(2)$*. In fact, once this is known, it is not hard to find global sections which are different from the local ones presented in reference [10], where the analysis carried out is thus not global. This result may contribute to clarify some aspects of the existence of the fidelity revivals discussed in [10].

Topological phases.- When the Berry connection A^m (0.4) on the Berry principal fiber bundle $U(E^m) \xrightarrow{\pi} B$ is *flat*[9] $dA - [A, A] = 0$, then Berry´s phase becomes **topological**. This means that the phase or holonomy which the final state gains depends only on the homotopy class of the parameter curve $b(t) \in B$ and not on its geometry any more . When the structure group is abelian, the flatness condition reads $dA = 0$[14]. Topological phases appear in various distinct areas[15], below we give two simple examples.

Spin on the plane and quantum Hall effect in graphene.- For $J = 1, 2$, consider the Hamiltonian $H_J(b) = \varepsilon\ \hat{s} \cdot \check{n}_b^J$ for $\hat{s}$ being the vector of $\hat{S}_i$ matrices corresponding to spin s, $\check{n}_b^J = -(cosJ\varphi_b, sinJ\varphi_b)$, with φ_b the angle giving the direction of the external parameter b on the xy plane $\mathbb{R}^2$ and ε a constant. This example with $J = 1$ follows straight forwardly from our first example (as done in [1]), but we study it independently bellow to illustrate our general construction of the bundle $U(E^m)$.

In this case, the largest submanifold $B \subset \mathbb{R}^2$ satisfying the given conditions $(i)(ii)$ is $B = \mathbb{R}^2 - \{\mathbf{0}\}$ since at $b = \mathbf{0}$, all eigenvalues collapse to only one with full degeneracy. From result (II) above, we know that the bundle $U(E^m)$ is trivializable[16]. The eigenstates are nondegenerated, so the Berry phase is abelian and the Berry connection (0.4) is

$$A^m(b) = i\ J\ s\ d\varphi_b$$

for which, clearly, $dA^m = 0$. Consequently, the Berry connection is *flat* and the associated Berry phase is *topological*. Explicitly, for a path $b(t)$ in B the phase reads

$$U_{geom}^m(t_f) = exp\left(-i2\pi s\ WN(b(t))\right)$$

where $WN(b(t))$ denotes the winding number of $b(t)$ around $(0, 0)$ in the $xy-$plane.

The case $J = 1$, describes a spin s interacting with a magnetic field b varying on the plane. So, when $WN(b(t)) = +1$, for *fermions* (half integer s) the above phase factor is -1 whereas for *bosons* (integer s) it is $+1$ independently of the $\hat{S}_i$ eigenvalue m. This reproduces the results of [1].

Finally, monolayer and bilayer *graphene Berry phases* correspond to $J = 1, 2$, respectively[17]. For this systems, the parameter b represents the carrier´s momentum within $2d$ graphene. When a magnetic field is present, carriers describe closed trajectories in the plane, so the parameter $b(t)$ describes a simple closed loop surrounding $(0, 0)$ in B after one carrier revolution. Then, the associated topological Berry phase factor is

$$U_{geom}^{J,m}(t_f) = exp\left(-i2J\pi s\right)$$

which, for $s = \frac{1}{2}$, is $(-1)^J$ as explained in [17].

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[13] This is the matrix of the transition function in a chosen basis $\left| w_+^j(\vec{\Omega}_0^+) \right\rangle$ of $E_{\vec{\Omega}_0^+}^0$, with $\vec{\Omega}_0^+$ the north pole of S^2.

[14] Moreover, for $U(1)$ bundles and $B \subset \mathbb{R}^3$ then the flatness condition takes the more familiar form $\nabla \times \vec{A} = 0$.

[15] J. Zak, Phys. Rev. Lett. 62, 2747 (1989); G. P. Mikitik, Yu. V. Sharlai Phys. Rev. Lett. 82, 2147 (1999); S.C. Althorpe J. Chem. Phys. 124, 084105 (2006); P. Milman, R. Mosseri, Phys. Rev. Lett. 90, 230403 (2003).

[16] Indeed, for a fixed m, the section $|m(b)\rangle = exp(i\varphi_b s)exp\left(i\varphi_b \hat{S}_z\right)|b_0, m_y\rangle$ is globally defined in B.

[17] K. S. Novoselov et al, Nature Physics 2, 177 - 180 (2006), doi:10.1038/nphys245.